# Topological pumping in superconducting wires with Majorana fermions

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We study adiabatic pumping in normal metal–superconductor hybrid systems when the superconductor is in symmetry class D and therefore exhibits a phase diagram with non-trivial regions associated with Majorana fermions. We show that topological quantization of the pumped charge occurs when the phase diagram has isolated trivial regions surrounded by topological ones. We propose a set-up where this physics emerges, which is based on a spin-orbit coupled superconducting nanowire in a tilted Zeeman field.

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*Introduction.*—A periodic modulation of two or more independent parameters in a quantum system may give rise to a dc flow of charge in the absence of any applied bias voltage. This transport mechanism is known as charge pumping [1–3] and it is termed adiabatic when the pumping period $T$ is much larger than any characteristic time scale of the system. If the system is coupled to leads the pumped charge can be expressed (ignoring electron interaction) in terms of the scattering matrix [1, 4]. Adiabatic pumping (AP) is of geometrical nature and it is related to the Berry phase accumulated during the cyclic evolution. In his pioneering work, Thouless [3] showed that quantized AP can occur through one-dimensional insulating systems as a consequence of the topological properties of the Hamiltonian, strictly related to the existence of gapless points enclosed by the adiabatic pumping cycle in parameter space [5]. The same topological argument holds even for a finite system attached to leads [6]. The recent discovery of topological insulators has also lead to a topological classification of spin pumps [7].

In this work we show how to realize topological AP through a normal - superconductor hybrid system. When time-reversal symmetry is broken (*i.e.* the superconductor is in symmetry class D [8]) the topological classification distinguishes two possible phases of the superconductor [9, 10], depending on whether the system supports or not mid-gap, Majorana, states. Since quasiparticle transport is strongly suppressed in a sufficiently long superconductor due to the superconducting gap, the pumped charge is obtained from the evolution in time of the normal ($r_{ee}$) and Andreev ($r_{he}$) components of the reflection matrix $r$ of the hybrid system as [11]

$$Q = \frac{e}{2\pi}\int_0^T \Im m \mathrm{Tr}\left[\frac{dr_{ee}(\tau)}{d\tau}r^\dagger_{ee}(\tau) - \frac{dr_{he}(\tau)}{d\tau}r^\dagger_{he}(\tau)\right]d\tau \ . \tag{1}$$

Adiabaticity is achieved when $T$ is much larger than the quasiparticle dwell time inside the superconductor, *i.e.* the time it takes before a quasiparticle is reflected or Andreev reflected. When the superconductor is in symmetry class D and the normal lead supports a single mode, the reflection matrix reduces to a single amplitude, *i.e.* a quasiparticle can undergo either a normal reflection or an Andreev reflection. Unitarity of the scattering matrix requires that the corresponding amplitude is a simple phase factor $e^{i\beta}$ [12, 13]. The pumped charge is then given by

$$Q = \pm\frac{e}{2\pi}\int_0^T \frac{d\beta}{dt}\,dt = \pm e\frac{\beta(T)-\beta(0)}{2\pi} = \pm ne \ , \tag{2}$$

where the upper (lower) sign is for normal (Andreev) reflection and $n$ is an integer since scattering amplitudes must be single-valued. If the phase diagram in the parameter space is simply connected and the pumping path can be continuously contracted to a point, the integer $n$ must be zero. On the contrary, if the phase diagram is non-simply connected and the pumping path, running in the topological phase, encloses a non-topological region, no restriction is enforced on $n$. Therefore the adiabatically pumped charge can, in principle, be topological, in the sense that any continuous deformation of the pumping path in parameter space does not change the integer $n$ [14]. In what follows we demonstrate that topological AP con be obtained in a (topological) superconductor, even relaxing the constraint of a single open channel in the lead by using a quantum point contact (QPC) to couple the lead to the superconductor. We also show that our results are robust against disorder.

*The model.*— The system we consider for a class-D superconductor is a spin-orbit coupled semiconductor nanowire in proximity to a $s$-wave superconductor and subjected to a Zeeman field [15]. In Refs. [16, 17] it has been shown that when the Zeeman field is aligned parallel to the wire (along, *e.g.*, the $x$-direction) and the pairing amplitude is uniform, the phase diagram as a function of chemical potential $\mu$ and Zeeman field $V_x$ can be divided into simply-connected regions characterized by a definite phase. On the other hand, different regions supporting the same phase touch at single points. If it were possible to expand such contact points into regions of finite area (see Fig. 1) we would have a topologically non-simply connected phase diagram, paving the way for topological quantized AP. This is possible by adding a small

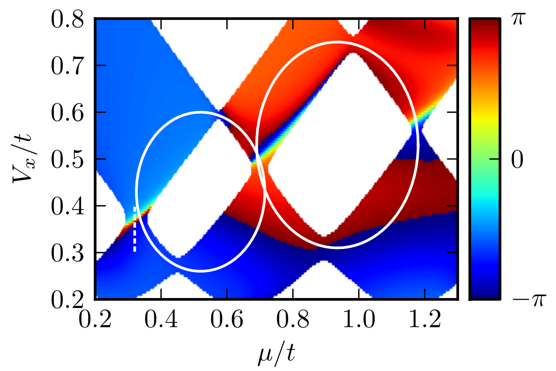


FIG. 1. (color online) Color plot of the phase of the Andreev reflection amplitude $r_{he}$ as a function of the longitudinal Zeeman field $V_x$ and the chemical potential $\mu$ in the superconducting region. Inside colored regions the superconductor is in the topologically non-trivial phase while inside white regions $r_{he} = 0$ and the superconductor is in the trivial phase. These results have been obtained with the following set of parameters: $W/a = 10$, $\alpha/t = 0.26$, and $V_z/t = 0.08$, where $a$ is the lattice constant. Inside the normal lead $\mu = V_x = 0$ so that only a single mode is propagating.

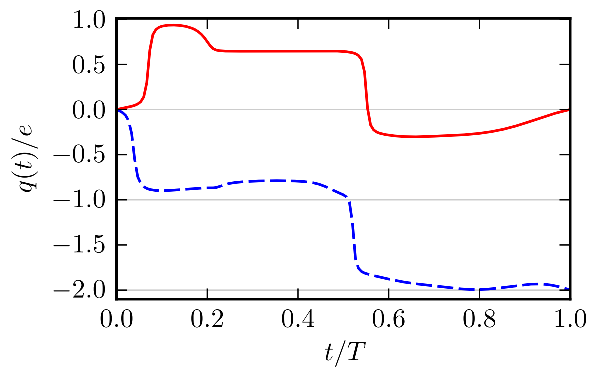


FIG. 2. (color online) Cumulative pumped charge $q(t)$ (in units of $e$) as a function of time $t$ (in units of the pumping period $T$). The thick solid line refers to the leftmost elliptic path in Fig. 1, while the thick dashed line to the rightmost one. At the end of the cycle, the final pumped charge $Q \equiv q(T)$ is quantized.

component of the Zeeman field in a direction perpendicular to the wire (say the $z$-direction), while allowing for a pairing amplitude that varies along the width of the nanowire [17, 18]. The latter situation is consistent with current experiments in which the proximity effect is induced by contact with a bulk superconductor from one side of the nanowire [19–21]. It is thus reasonable to assume that the amplitude of the superconducting pairing leaking in the nanowire decreases as we move away from the side in contact with the superconductor. We have considered different profiles of the pairing amplitude along the transverse $y$-direction (with $0 \le y \le W$, $W$ being the width of the wire) [22].

The Hamiltonian of the superconducting nanowire then reads

$$\hat{\mathcal{H}} = \sum_{i,j,\sigma,\sigma'} h_{ij,\sigma,\sigma'} \hat{c}^\dagger_{i,\sigma} \hat{c}_{j,\sigma'} + \sum_i [\Delta(y_i)\ \hat{c}^\dagger_{i,\uparrow} \hat{c}^\dagger_{i,\downarrow} + \text{H.c.}]$$

with

$$h_{ij} = [(4t-\mu)\,\mathbb{1} + \boldsymbol{V}\cdot\boldsymbol{\sigma}]\,\delta_{ij} - [t\,\mathbb{1} - i\alpha(\nu'_{ij}\sigma^x - \nu_{ij}\sigma^y)]\,\delta_{\langle i,j\rangle}\ ,$$

where $c^\dagger_{i,\sigma}$ ($c_{i,\sigma}$) creates (destroys) an electron at site $i$ with spin $\sigma$, $\delta_{\langle i,j\rangle}$ restricts sites $i$ and $j$ to be nearest-neighbors, $\sigma^b$ (with $b = x, y, z$) are spin-1/2 Pauli matrices, $\nu_{ij} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{d}}_{ij}$, and $\nu'_{ij} = \hat{\boldsymbol{y}} \cdot \hat{\boldsymbol{d}}_{ij}$, with $\hat{\boldsymbol{d}}_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j)/|\boldsymbol{r}_i - \boldsymbol{r}_j|$ being the unit vector connecting site $j$ to site $i$. The parameters involved in $\hat{\mathcal{H}}$ are the hopping energy $t$, the spin-orbit coupling strength $\alpha$, and the chemical potential $\mu$. The only non-zero components of the Zeeman field are $V_x$ and $V_z$.

*Pumping in a single-mode lead.*—We first consider the situation in which the normal lead coupled to the superconducting nanowire supports a single open channel. The topological invariant $\mathcal{N}$ distinguishing topologically trivial from non-trivial phases is related to the reflection matrix $r$ at the normal-metal/superconductor (NS) interface according to $\mathcal{N} = \det r$ [23]. For a single-mode normal lead this reduces to $\mathcal{N} = |r_{ee}|^2 - |r_{he}|^2$. When the superconductor is in the topological phase ($\mathcal{N} = -1$), normal reflection is absent, $r_{ee} = 0$, and the mode is perfectly Andreev-reflected by the Majorana fermion at the NS boundary, $|r_{he}| = 1$ [13]. On the contrary, when the superconductor is in the trivial phase ($\mathcal{N} = +1$), the mode is fully reflected ($|r_{ee}| = 1$). We have evaluated the reflection matrix by using a method which combines recursive Green's function algorithms [24] with a wave-function matching approach [25]. In Fig. 1 the superconductor is in the non-trivial phase ($|r_{he}| = 1$) inside colored regions and the color coding depicts the phase of the Andreev reflection amplitude as a function of $V_x$ and $\mu$ in the superconducting region. Inside the white regions the superconductor is in the topologically trivial phase and $r_{he}$ vanishes. At the boundary of such white regions the gap closes, signaling a quantum phase transition between the topologically trivial and non-trivial phases. As a consequence, the topological sector of the phase diagram is non-simply connected and the elliptic paths marked by white solid lines in Fig. 1 can not be shrunk to a point without crossing the phase boundary. The phase of $r_{he}$ is almost constant throughout the parameter space with the exception of stripes connecting different non-topological (white) regions, across which the phase winds by approximately $2\pi$. We thus expect that when the pumping path coincides with non-contractible elliptic paths of the type shown in Fig. 1 these resonant stripes contribute by roughly $e$ to the pumped charge $Q$.

To confirm this expectation we have computed the adiabatic pumped charge along both paths shown in Fig. 1

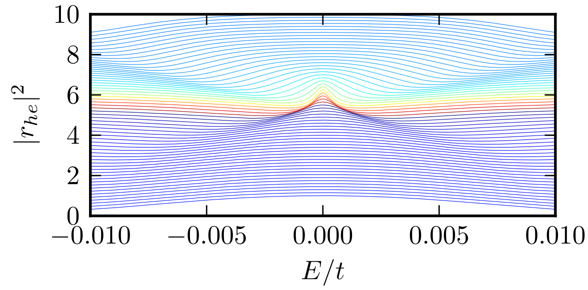


FIG. 3. (color online) Andreev reflection $|r_{he}|^2$ as a function of energy $E$ for several increasing values of the longitudinal Zeeman field, $0.3 \leq V_x/t \leq 0.42$, at fixed $\mu/t = 0.32$ (see dashed line in Fig. 1). Different curves have been offset for clarity. The color coding is associated with the phase of $r_{he}$ at zero energy as in Fig. 1.

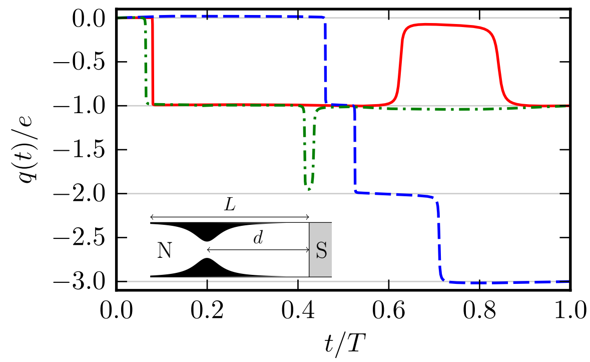


FIG. 4. (color online) Cumulative pumped charge $q(t)$ (in units of $e$) as a function of time $t$ (in units of the pumping period $T$) in the presence of a QPC at a distance $d = 60\ a$ from the NS interface. At the end of the cycle the final pumped charge $Q \equiv q(T)$ is quantized. Results for different trajectories in parameter space are denoted with distinct line styles. The following set of parameters has been adopted: $W/a = 10$, $\alpha/t = 0.15$ and $V_z/t = 0.1$. In the inset we have sketched the setup with a QPC at a distance $d$ from the NS boundary in order to restrict the number of propagating modes. In simulations with disorder, random potential fluctuations are considered in a region $L > d$ inside the normal lead before the NS interface.

by using a gauge-invariant discretized form of Eq. (1). We modeled the evolution in time of the pumping parameters as follows

$$\begin{cases} \mu(t) = \bar{\mu} + \delta\mu \cos(2\pi\, t/T) \\ V_x(t) = \bar{V}_x + \delta V_x \sin(2\pi\, t/T) \end{cases}, \tag{3}$$

where $\bar{\mu}/t = 0.52$, $\delta\mu/t = 0.195$, $\bar{V}_x/t = 0.43$, and $\delta V_x/t = 0.17$ for the left path, while $\bar{\mu}/t = 0.935$, $\delta\mu/t = 0.245$, $\bar{V}_x/t = 0.53$, and $\delta V_x/t = 0.22$ for the right one. Experimentally this evolution could be realized by properly designing metal gates and coils to introduce time-dependent gate voltages and magnetic fields acting only on the superconducting section of the nanowire.

In Fig. 2 we show our numerical results for the cumulative pumped charge $q(t)$ defined as in Eq. (1) with the integral running from 0 to $t$. The solid (dashed) line refers to the leftmost (rightmost) path in Fig. 1. The total pumped charge in a cycle $Q \equiv q(T)$ is strictly quantized and equal to 0 and $-2\,e$, respectively, in agreement with the general picture described above. In both cases the leading contribution to $Q$ during the cycle comes from the resonant stripes that connect neighboring phase boundaries. Along the rightmost path both resonant stripes give rise to a counterclockwise rotation of the phase of $r_{he}$ so that their corresponding contributions sum up to $-2e$. Along the leftmost path the winding of the Andreev reflection amplitude phase across the resonant stripes is opposite and the total pumped charge vanishes. It should be clear that the value of the pumped charge is topological, *i.e.* it does not change under smooth deformations of the pumping path in parameter space. Indeed, it is not possible to deform the contour in order to avoid the crossing of the resonant stripes without intersecting the phase boundary lines.

We have verified that similar resonant stripes are present also in different parameter ranges or with different profiles of the pairing amplitude, eventually giving rise to a topological pumping. We have noticed that $Q/e$ can assume any integer value, even or odd, depending on the details of the system under consideration. We stress that the presence of the resonant stripes is strictly connected to the non-trivial topology of the phase diagram, and thus to the existence of two different phases of the superconductor, one of them supporting Majorana fermions.

To further support the importance of Majorana fermions in the pumping mechanism, we observe that the resonant stripes are associated with a narrowing of the Majorana-induced zero-energy resonance in the Andreev reflection, distinctive of the non-trivial phase [26, 27]. This is shown in Fig. 3 where we plot $|r_{he}|^2$ (proportional to the differential conductance of the NS interface) as a function of energy. Different curves are reported (and offset for clarity) for several values of $V_x$ at fixed $\mu/t = 0.32$, corresponding to points along the dashed segment in Fig. 1.

*QPC geometry.*—We now show that topological AP can be achieved in a more realistic situation in which the number of open channels in the normal electrode is arbitrary (so that $r_{ee}$ and $r_{he}$ are now matrices) but a QPC restricts the number of modes incident on the NS boundary (see inset in Fig. 4). The QPC consists of a saddle-point constriction [13, 28] at a distance $d$ from the NS interface (see Fig. 4). When the pinch-off potential of the QPC is chosen so that only a single mode is transmitted, it is possible to prove (we assume the superconductor to be in the topological phase) that the matrix $r_{he}$ changes in time only through a global phase factor

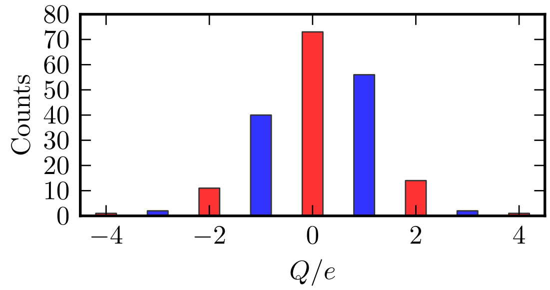


FIG. 5. (color online) Histogram of the pumped charge $Q/e$ obtained for 100 disorder realizations with $L/a = 200$, $d/a = 140$, and $U_{\rm dis}/t = 0.5$. Light gray (red) and dark gray (blue) bars refer to two different sets of parameters and distinct pumping paths. We notice that although the value of $Q$ depends on the disorder configuration, it is always quantized and its parity is preserved.

while $r_{ee}$ remains constant, so that the pumped charge in a cycle is still quantized.

This is confirmed by the numerical results in Fig. 4, which show the charge pumped along three different non-contractible paths. The pumping parameters $V_x$ and $\mu$ in the superconductor follow the time evolution in Eq. (3), while in the normal lead we fix $V_x \equiv \bar{V}_x$ and $\mu \equiv \bar{\mu}$. The three curves in Fig. 4 simply differ in the values of $\bar{V}_x$, $\delta V_x$, $\bar{\mu}$, and $\delta\mu$, which identify the pumping paths through Eq. (3).

The pumped charge $Q$ is quantized to integer multiples of $e$. As in the case of a single-mode lead, the value of $Q$ can be interpreted as the result of contributions from the crossing of resonant stripes which gives rise to a twist of the global phase of $r_{he}$ by $\sim 2\pi$ (data not shown). Each time the system crosses a resonant stripe approximately a charge $e$ is pumped in the normal lead, corresponding to the step-like features in Fig. 4. When the distance $d$ is varied, the interference pattern changes in such a way that resonant lines could come in/out of the way of the pumping path. We have noticed that, although the value of $Q/e$ can vary, it is always strictly quantized to integer values and the parity of $Q/e$ is conserved. This means that single resonant lines cannot appear or disappear along the pumping cycle but they can only be created/destroyed in pairs or change the sign of the corresponding phase winding.

We have also investigated the effect of disorder by introducing a random on-site potential in a region of length $L$ ($> d$) on the left of the NS interface (see Fig. 4). Fluctuations of the on-site energy lie in the interval $[-U_{\rm dis}/2, U_{\rm dis}/2]$. We have noticed that quantization of the pumped charge is almost exactly preserved up to large values of the disorder strength ($U_{\rm dis}/t \approx 5-6$) provided that a single mode is transmitted through the QPC (see Fig. 4). As shown in Fig. 5, the details of a specific disorder configuration affect the value of the pumped charge but, given the system parameters and the pumping path, the parity of $Q/e$ is insensitive to smooth changes of the Hamiltonian.

*Conclusions* – We have shown that topological adiabatic pumping occurs in a class-D superconducting nanowire connected to a metallic lead, provided that a single mode of the latter is affected by the presence of Majorana fermions present at the endpoints of the superconducting nanowire. This is the case, for example, when the lead supports a single propagating mode or when the nanowire is coupled to the lead through a quantum point contact. The topological nature of pumping consists in the fact that any continuous deformation of the pumping path in parameter space does not change the charge pumped in a cycle. The necessary condition to achieve a finite quantized value of the pumped charge is that the phase diagram presents a non-simply connected structure, where isolated non-topological regions are surrounded by connected topological ones. This is possible by allowing both a non-uniform pairing amplitude and a tilted Zeeman field. Non-contractible pumping paths in parameter space can thus be identified within the topological phase. We have furthermore verified that the quantization of the pumped charge is robust against disorder.

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$$\Delta_{\rm pow}(y) = \Delta_0 \left[1 - \theta \left(y/W\right)^{10}\right]$$

and

$$\Delta_{\rm exp}(y) = \Delta_0 \left\{1 - \frac{\theta}{2}\left[1 + \tanh\left(8(y/W - 1/2)\right)\right]\right\} ,$$

where $0 \leq \theta \leq 1$ is a dimensionless parameter which measures the suppression of the superconducting pairing across the wire [18].

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